Problem Set 4

This fourth problem set explores functions, propositional logic, and first-order logic. It will be the final problem set in our exploration of discrete mathematics, and by the time you're done with it you'll be ready to start tackling the limits of computability!

As always, please feel free to drop by office hours or send us emails if you have any questions. We'd be happy to help out.

This problem set has 150 possible points. It is weighted at 7% of your total grade.

Good luck, and have fun!

Checkpoint due Monday, October 21 at 2:15PM

Assignment due Friday, October 25 at 2:15PM

Write your solutions to the following problems and submit them by Monday, October 21ˢᵗ at the start of class. These problems will be graded on a 0/12/25 scale based on whether you have attempted to solve all the problem, rather than on correctness. We will try to get these problems returned to you with feedback on your proof style this Wednesday, October 23ʳᵈ.

Checkpoint Problem: Paradoxical Sets (25 Points if Submitted)

What happens if we take *absolutely everything* and throw it into a set? If we do, we would get a set called the *universal set*, which we denote $\mathscr{U}$:

$$\mathscr{U} = \{\, x \mid x \text{ exists} \,\}$$

Absolutely everything would belong to this set: $1 \in \mathscr{U}$, $\mathbb{N} \in \mathscr{U}$, philosophy $\in \mathscr{U}$, CS103 $\in \mathscr{U}$, etc. In fact, we'd even have $\mathscr{U} \in \mathscr{U}$, which is strange but not immediately a problem.

Unfortunately, the set $\mathscr{U}$ doesn't actually exist, as its existence would break mathematics.

i.   Prove that if A and B are sets where $A \subseteq B$, then $|A| \leq |B|$. Although this probably makes intuitive sense, to formally prove this result, find an injection $f : A \to B$ and prove that your function is injective.

ii. Using your result from (i), prove that if $\mathscr{U}$ exists at all, then $|\wp(\mathscr{U})| \leq |\mathscr{U}|$.

iii. Using your result from (ii) and Cantor's Theorem, prove that $\mathscr{U}$ cannot exist.

The result you've proven shows that there is a collection of objects (namely, the collection of everything that exists) that cannot be put into a set. This goes against our intuition of what a set is. When this was discovered at the start of the twentieth century, it caused quite a lot of chaos in the math world and led to a reexamination of logical reasoning itself and a more formal definition of what objects can and cannot be gathered into a set. If you're curious to learn more about what sets can and cannot be created, take Math 161 (Set Theory).

Problem One: Cartesian Products and Cardinalities (20 Points)

The cardinality of the Cartesian product of two sets depends purely on the cardinalities of those sets, not on what the elements of those sets actually are. This question will ask you to prove this.

Let A, B, C, and D be sets where $|A| = |C|$ and $|B| = |D|$. Our goal is to prove $|A \times B| = |C \times D|$. Since we know $|A| = |C|$, there has to be some bijection $g : A \to C$. Since we know $|B| = |D|$, there has to be some bijection $h : B \to D$. Now, consider the function $f : A \times B \to C \times D$ defined as follows:

$$f(a, b) = (g(a), h(b))$$

  i. Using the function f defined above, prove that if A, B, C, and D are sets where $|A| = |C|$ and $|B| = |D|$, then we have $|A \times B| = |C \times D|$. Specifically, prove that f is a bijection between $A \times B$ and $C \times D$.

We can define the "Cartesian power" of a set as follows. For any set A and any positive natural number n, we define A^n inductively:

$$A^1 = A$$

$$A^{n+1} = A \times A^n \text{ (for } n \geq 1)$$

  ii. Using your result from (i) and the above definition, prove that $|\mathbb{N}^k| = |\mathbb{N}|$ for all nonzero $k \in \mathbb{N}$. This result means that for any nonzero finite k, there are the same number of k-tuples of natural numbers as natural numbers. *(Hint: You might want to use a result from lecture.)*

(Here's some justification for the problem you just did. You don't need to read this if you don't want to, but we think you might find it interesting. ☺)

In lecture, we defined cardinality on a relative basis by showing how to define what the statements $|A| = |B|$, $|A| \neq |B|$, $|A| \leq |B|$, and $|A| < |B|$ mean in terms of functions between those sets. We specifically didn't talk about cardinalities as actual quantities, since it's tricky to do so. That said, we can actually talk about set cardinalities as actual objects. Intuitively speaking, the cardinality of a set is a measure of how large that set is. For finite sets, the cardinality of that set will be a natural number, and for infinite sets the cardinality of that set will be an *infinite cardinality*, a generalization of the natural numbers that measure the sizes of infinite sets. For example, $\aleph_0$, which we introduced in our first lecture as the cardinality of $\mathbb{N}$, is an infinite cardinality.

Given cardinalities κ_1 and κ_2, we define the product of those two cardinalities, $\kappa_1 \cdot \kappa_2$, to be $|A \times B|$, where A and B are any sets where $|A| = \kappa_1$ and $|B| = \kappa_2$. For example, $4 \cdot 3$ is *by definition* the cardinality of $|A \times B|$ for any set A of cardinality 4 and any set B of cardinality 3. Similarly, *by definition* the value of $\aleph_0 \cdot \aleph_0$ is the cardinality of $|A \times B|$ for any sets A and B of cardinality $\aleph_0$.

To make sure that this definition is legal, we have to make sure that the cardinality of the Cartesian product depends purely on the cardinalities of the two sets, not their contents. For example, this definition wouldn't give us a way to compute $4 \cdot 3$ if the cardinality of the Cartesian product of a set of four apples and three oranges was different than the cardinality of the Cartesian product of a set of four unicorns and three ponies. We need to show that for any sets A, B, C, and D, that if $|A| = |C|$ and $|B| = |D|$, then $|A \times B| = |C \times D|$. That way, when determining the value of $\kappa_1 \cdot \kappa_2$, it doesn't matter which sets of cardinality κ_1 and κ_2 we pick; any choice works. Your proof from (i) filled in this step.

Your result from (ii) shows that $\aleph_0^n = \aleph_0$ for any positive natural number n. Isn't infinity weird?

Problem Two: Simplifying Cantor's Theorem? (8 Points)

In lecture, we proved Cantor's theorem, that $|S| < |\wp(S)|$ for any set S. In order to do so, we used a diagonal argument to show that $|S| \neq |\wp(S)|$.

Below is a purported proof that $|S| \neq |\wp(S)|$ that doesn't use a diagonal argument:

> *Theorem:* For any set S, we have $|S| \neq |\wp(S)|$.
>
> *Proof:* Let S be any set and consider the function $f : S \rightarrow \wp(S)$ defined as $f(x) = \{x\}$. To see that this is a valid function from S to $\wp(S)$, note that for any $x \in S$, we have $\{x\} \subseteq S$. Therefore, $\{x\} \in \wp(S)$ for any $x \in S$, so f is a legal function from S to $\wp(S)$.
>
> Let's now prove that f is injective. Consider any $x_1, x_2 \in S$ where $f(x_1) = f(x_2)$. We'll prove that $x_1 = x_2$. Because $f(x_1) = f(x_2)$, we have $\{x_1\} = \{x_2\}$. Since two sets are equal iff their elements are the same, this means that $x_1 = x_2$, as required.
>
> However, f is not surjective. Notice that $\varnothing \in \wp(S)$, since $\varnothing \subseteq S$ for any set S, but that there is no x such that $f(x) = \varnothing$; this is because $\varnothing$ contains no elements and $f(x)$ always contains one element. Since f is not surjective, it is not a bijection. Thus $|S| \neq |\wp(S)|$ ∎

Unfortunately, this proof is incorrect and doesn't prove $|S| \neq |\wp(S)|$. What's wrong with this proof? Justify your answer.

Problem Three: Understanding Diagonalization (8 Points)

In lecture, we proved that $|\mathbb{N}| \neq |\mathbb{R}|$ by showing that for any function $f : \mathbb{N} \rightarrow \mathbb{R}$, it's always possible to find some real number $d \in \mathbb{R}$ such that $f(n) \neq d$ for all $n \in \mathbb{N}$. This was our first proof by diagonalization, and it's definitely more complex than the other proofs we've done so far.

This problem asks you some questions about that proof so that you get a better understanding of how the proof works. We hope that this helps clear up any lingering doubts you might have about how the proof works.

Given a function $f : \mathbb{N} \rightarrow \mathbb{R}$, we defined the diagonal number d by giving its decimal representation as follows:

$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 0 \\ 0 & \text{otherwise} \end{cases}$$

Here, $r[0]$ is the integer part of r, and $r[n]$ is the nth decimal digit of r if $n > 0$.

i.   Suppose that $f(n) = n$. What is the diagonal number d in this case?

ii. Let d_0 be the value of d that you found in part (i). Give an example of a function $g : \mathbb{N} \rightarrow \mathbb{R}$ such that there is an $n \in \mathbb{N}$ where $g(n) = d_0$.

Your result from (ii) clarifies an important point about the diagonalization proof. Any individual function from $\mathbb{N}$ to $\mathbb{R}$ has to leave some real number uncovered, but there isn't a single real number that's left uncovered by all functions.

iii. What is the diagonal number for your function g that you defined in part (ii)? *(Note: if you're having trouble seeing what the diagonal number is, you might have made your function g more complicated than it needs to be. You can pick a very simple function g that satisfies the necessary requirements.)*

Problem Four: Uncomputable Functions (28 Points)

(We strongly suggest completing Problems Two and Three before attempting this problem.)

There can be many functions from one set A to a second set B. This question explores how many functions of this sort there are. Consider the following set:

$$\mathbb{N}^{\mathbb{N}} = \{ h \mid h : \mathbb{N} \to \mathbb{N} \}$$

That is, $\mathbb{N}^{\mathbb{N}}$ is the set of all functions whose domain and codomain are $\mathbb{N}$. Note that $\mathbb{N}^{\mathbb{N}}$ does *not* mean "$\mathbb{N}$ raised to the $\mathbb{N}$th power." It's just the notation we use for set of all functions from $\mathbb{N}$ to itself. This is a *very* large set, and in this problem you'll see just how large it is.

First, we're going to ask you to prove that $|\mathbb{N}| \le |\mathbb{N}^{\mathbb{N}}|$. This means that you will need to find an injective function $g : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$. To better understand what means, given any natural number n, the value of $g(n)$ is itself a function. That function takes a natural number as input and then produces a natural number.

Notationally, if you're trying to define a function $g : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$, you can write

$$g(a)(b) = c$$

to mean "the function returned by $g(a)$ is a function that, given input b, evaluates to c."

i. Define an injective function $g : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$. That is, find a function from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$ such that given any two different natural numbers as inputs, you will always get two different functions as outputs. You don't need to prove that your function is an injection just yet – we just want you to tell us what function you're picking.

ii. Prove the function you came up with in part (i) is an injection. You might want to use the fact that two functions $h_1 : \mathbb{N} \to \mathbb{N}$ and $h_2 : \mathbb{N} \to \mathbb{N}$ are equal iff $h_1(n) = h_2(n)$ for all $n \in \mathbb{N}$. Equivalently, this means that $h_1 \ne h_2$ iff there is some n where $h_1(n) \ne h_2(n)$.

Now, we are going to ask you to prove that $|\mathbb{N}| \ne |\mathbb{N}^{\mathbb{N}}|$ using a proof by diagonalization. The diagonalization proofs we did in lecture relied on defining some "diagonal object" d in response to an alleged bijection between two sets. Before you start writing your full proof that $|\mathbb{N}| \ne |\mathbb{N}^{\mathbb{N}}|$, we want you to start off by defining the "diagonal function" you're going to use.

iii. Let $f : \mathbb{N} \to \mathbb{N}^{\mathbb{N}}$ be an arbitrary function from $\mathbb{N}$ to $\mathbb{N}^{\mathbb{N}}$. Create and describe a "diagonal function" $d : \mathbb{N} \to \mathbb{N}$ such that $f(n) \ne d$ for any $n \in \mathbb{N}$. In other words, the function d you choose should be different from every function that can be produced as the output of $f(n)$. Then, briefly describe (but do not formally prove) why your diagonal function d must be different from $f(n)$ for all $n \in \mathbb{N}$.

As hints on this problem:

1. You might find it useful to draw out a picture to figure out what your diagonal function will be. However, you should give a precise mathematical definition of what d in your answer.

2. Think back to the intuition behind diagonalization. You'll probably want to build d out of infinitely many "pieces" such that the 0th "piece" of d is different from the 0th "piece" of $f(0)$, the 1st "piece" of d is different from the 1st "piece" of $f(1)$, etc.

iv. Using the diagonal function d you defined in part (iii), prove that $|\mathbb{N}| \ne |\mathbb{N}^{\mathbb{N}}|$. Make sure that you formally prove why your diagonal function has the property that $f(n) \ne d$ for any $n \in \mathbb{N}$.

The result you have just proven shows that $\aleph_0 < \aleph_0^{\aleph_0}$, even though $\aleph_0 = \aleph_0^n$ for any positive integer n. Again, isn't infinity weird?

Problem Five: Propositional Equivalences (12 Points)

For each of the following pairs of propositional formulas, decide whether those formulas are equivalent. If they are, write truth tables for each to show that they are the same. If they aren't, find an assignment of true and false to their propositional variables and show that one evaluates to true while the other evaluates to false.

   i. $p \rightarrow q$ and $\neg q \rightarrow \neg p$

   ii. $\neg(p \rightarrow q)$ and $p \rightarrow \neg q$

   iii. $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$

   iv. $p \lor q$ and $\neg p \rightarrow q$

Problem Six: First-Order Negations (16 points)

For each of the first-order logic formulas below, find a first-order logic formula that is the negation of the original statement. Your final formula must not have any negations in it, except for direct negations of predicates. For example, the negation of

$$\forall x. \ (p(x) \rightarrow \exists y. \ (q(x) \land r(y)))$$

would be found by pushing the negation in from the outside as follows:

$$\neg(\forall x. \ (p(x) \rightarrow \exists y. \ (q(x) \land r(y))))$$
$$\exists x. \ \neg(p(x) \rightarrow \exists y. \ (q(x) \land r(y)))$$
$$\exists x. \ (p(x) \land \neg\exists y. \ (q(x) \land r(y)))$$
$$\exists x. \ (p(x) \land \forall y. \ \neg(q(x) \land r(y)))$$
$$\exists x. \ (p(x) \land \forall y. \ (q(x) \rightarrow \neg r(y)))$$

Show every step of the process of pushing the negation into the formula (along the lines of what is done above). You don't need to formally prove that your negations are correct.

   i. $\exists S. \ (Set(S) \land \forall x. \ x \notin S)$

   ii. $\forall p. \ \forall q. \ (p \in \mathbb{N} \land q \in \mathbb{N} \land q \neq 0 \rightarrow q \times q \times 2 \neq p \times p)$

   iii. $\forall x \in \mathbb{R}. \ \forall y \in \mathbb{R}. \ (x < y \rightarrow \exists q \in \mathbb{Q}. \ (x < q \land q < y))$

   iv. $(\forall x. \ \forall y. \ \forall z. \ (R(x, y) \land R(y, z) \rightarrow R(x, z))) \rightarrow (\forall x. \ \forall y. \ \forall z. \ (R(y, x) \land R(z, y) \rightarrow R(z, x)))$

Problem Seven: Translating into Logic (28 points)

In each of the following, you will be given a list of first-order predicates and functions along with an English sentence. In each case, write a statement in first-order logic that expresses the indicated sentence. Your statement may use any first-order construct (equality, connectives, quantifiers, etc.), but you **must** only use the predicates, functions, and constants provided.

i. Given the predicate

   Natural(x), which states that x is an natural number;

   the function

   Product(x, y), which yields the product of x and y;

   and the constants 1 and 137, write a statement in first-order logic that says "137 is prime."

ii. Given the predicates

   Morality(m), which states that m is a morality;
   Practice(m), which states that m is practiced; and
   Preach(m), which states that m is preached;

   write a statement in first-order logic that says "there are exactly two moralities: one of which is practiced but not preached, and one of which is preached but not practiced" (paraphrased from a quote by Bertrand Russell).

iii. Given the predicates

   $x \in y$, which states that x is an element of y, and
   Set(S), which states that S is a set,

   write a statement in first-order logic that says "every set has a power set."

iv. Given the predicates

   Lady(x), which states that x is a lady;
   Glitters(x), which states that x glitters;
   IsSureIsGold(x, y), which states that x is sure that y is gold;
   Buying(x, y), which states that x buys y; and
   StairwayToHeaven(x), which states that x is a Stairway to Heaven;

   write a statement in first-order logic that says "There's a lady who's sure all that glitters is gold, and she's buying a Stairway to Heaven."*

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* Let's face it – the lyrics to Led Zeppelin's "Stairway to Heaven" are impossible to decipher. Hopefully we can gain some insight by translating them into first-order logic!

Problem Eight: Course Feedback (5 Points)

We want this course to be as good as it can be, and we'd really appreciate your feedback on how we're doing. For a free five points, please answer the following questions. We'll give you full credit no matter what you write (as long as you write something!), but we'd appreciate it if you're honest about how we're doing.

i. How hard did you find this problem set? How long did it take you to finish? Does that seem unreasonably difficult or time-consuming for a five-unit class?

ii. Did you attend a recitation section this week? If so, did you find it useful?

iii. How is the pace of this course so far? Too slow? Too fast? Just right?

iv. Is there anything in particular we could do better? Is there anything in particular that you think we're doing well?

Extra Credit Problem: The Diagonal Number (5 Points Extra Credit)

Suppose that we change the diagonal number from the proof that $|\mathbb{N}| \neq |\mathbb{R}|$ to be the following:

$$d[n] = \begin{cases} 1 & \text{if } f(n)[n] = 9 \\ 9 & \text{otherwise} \end{cases}$$

Interestingly, if we define the diagonal number this way, the proof by diagonalization fails.

Explain why the proof fails with this modified diagonal number. Give a specific example that demonstrates the error in the proof.